Simple Finite-Period Models on Sovereign Debt and Default

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March 12, 2015

Introduction

- In this class we will look at a couple of simple models on sovereign default that we can characterize analytically

- The objective is to get some insight into how these models operate and generate default, and understand what dimensions are potentially interesting from both an empirical and theoretical standpoint

- In particular, we will characterize both two and three-period models with and without default in the tradition of Eaton and Gersovitz (1981)

Some Lessons from Giancarlo's International Finance Class on Sovereign Risk

- To avoid overlap with Giancarlo's international finance lectures on sovereign risk, I will discuss topics (debt duration, multiplicity) not covered in that class
- But given that the issues he discussed are important, and not all students attend both classes, let me summarize some of the key lessons from his class
- Default is not just about ability to pay; it is also about willingness to pay (concept: enforceability)
- This matters especially as states of the world underlying contracts are not verifiable
- How well do international financial markets function in the presence of default?

Some Lessons from Giancarlo's International Finance Class on Sovereign Risk

- On sanctions ... with high enough sanctions, perfect risk sharing is attainable as severe punishment induces country to never default
- On reputation ... default only when gains from not repaying today outweigh costs of reduced access to financial markets in the future
- Bulow Rogoff ... lenders need ability to seize borrower assets or impose default cost otherwise default and invest elsewhere

Some Lessons from Giancarlo's International Finance Class on Sovereign Risk

- On secondary markets . . . favoring domestic lenders lmits ability of borrower to renege on promise to repay debt
- On debt swaps . . . swapping old debt for more senior new debt can be benefiical to borrowing governments
- On debt forgiveness . . . beneficial for creditors only if on right side of Laffer curve so it affects borrowers' default decision and probability that the state of the economy will be good tomorrow (does not just reduce payments in the good state)

Outline for the Rest of This Class

- Two period model with complete markets, incomplete markets with no default, incomplete markets with default
- Three period model with short and long bonds, with short bond only, with long bond only
- Fundamental and non-fundamental risk and their relation to debt duration

Two-Period Model

- Consider a two-period small open economy
- Its representative consumer has quadratic expected utility over consumption
- Income in the first period is 1 with certainty
- Income in the second period is 4 with probability 1/2 and 2 with probability 1/2
- Consumer trades with risk-neutral international investors (that value equally period 1 and 2 consumption)

Two-Period Model: Complete Markets

- Suppose the consumer can trade Arrow securities contingent on the realization of period 2 income
- What will equilibrium allocations and prices look like?
- The problem of the borrower can be written as follows

$$\max U = u(c_0) + \frac{1}{2}u(c_{1h}) + \frac{1}{2}u(c_{1l})$$

$$c_0 = 1 + q_h p_h + q_l p_l$$

$$c_{1h} = 4 - p_h$$

$$c_{1l} = 2 - p_l$$

Two-Period Model: Complete Markets

- Given quadratic utility, marginal utility is linear in consumption, and the solution is interior
- Further, with risk-neutral lenders and no default, we know

$$q_l = q_h = 1/2$$

- We then have the following equilibrium conditions:

$$u'(c_0) = \lambda, \qquad \frac{1}{2}\lambda = \mu_H$$

$$\frac{1}{2}u'(c_{1h}) = \mu_H, \qquad \frac{1}{2}\lambda = \mu_L$$

$$\frac{1}{2}u'(c_{1l}) = \mu_L$$

Two-Period Model: Complete Markets

- This yields

$$c_0 = c_{1h} = c_{1l} = \bar{c}$$

- Which then implies

$$p_h = p_l + 1 \Rightarrow p_l = 0, p_h = 2, \bar{c} = 2$$

- Perfect risk-sharing is achieved!

Two-Period Model: Complete Markets

The equilibrium with complete markets is then given by

$$q_h = \frac{1}{2}$$
$$q_l = \frac{1}{2}$$
$$p_h = 2$$
$$p_l = 0$$
$$c_0 = 2$$
$$c_{1h} = 2$$
$$c_{1l} = 2$$

Two-Period Model: Incomplete Markets, No Default

- Now suppose instead that the agent can only trade a non-contingent bond
- What is the equilibrium interest rate and consumption in each period and each state?
- The problem of the borrower now becomes

$$\max U = u(c_0) + \frac{1}{2}u(c_{1h}) + \frac{1}{2}u(c_{1l})$$

$$c_0 = 1 + qp$$

$$c_{1h} = 4 - p$$

$$c_{1l} = 2 - p$$

Two-Period Model: Incomplete Markets, No Default

- Given quadratic utility, marginal utility is linear in consumption, and the solution is interior
- Further, with risk-neutral lenders and no default, the promise is always repaid, hence
$$q = 1$$
- We then have the following equilibrium conditions:

$$u'(c_0) = \lambda,$$
$$\frac{1}{2}u'(c_{1h}) = \mu_H$$
$$\frac{1}{2}u'(c_{1l}) = \mu_L$$
$$\lambda = \mu_H + \mu_L$$

Two-Period Model: Incomplete Markets, No Default

- This yields
$$u'(c_0) = \frac{1}{2}u'(c_{1h}) + \frac{1}{2}u'(c_{1l})$$

- Linearity of marginal utilities from quadratic utility imply
$$c_0 = \frac{1}{2}c_{1h} + \frac{1}{2}c_{1l}$$
$$1 + p = \frac{1}{2}(4 - p) + \frac{1}{2}(2 - p)$$
$$p = 1 \Rightarrow c_0 = 2, c_{1h} = 3. c_{1l} = 1$$

- No longer perfect risk-sharing, but some insurance

Two-Period Model: Incomplete Markets, No Default

The equilibrium with incomplete markets and no default is given by

$$q_l = 1$$
$$p_l = 1$$
$$c_0 = 2$$
$$c_{1h} = 3$$
$$c_{1l} = 1$$

Two-Period Model: Incomplete Markets, Low Cost of Default

- Now assume that while the agent can still only trade a non-contingent bond, it can default on the bond in period 2
- If the borrower defaults, his income in period 2 falls to 1.5
- This is similar to the specification in Arellano (2008) where output is capped in periods where agents default
- Cost of default here is relatively low: output only falls from 2 to 1.5 in the low state
- What is the equilibrium interest rate and consumption in each period and each state?

Two-Period Model: Incomplete Markets, Low Cost of Default

- The problem for the borrower now becomes

$$\max U = u(c_0) + \frac{1}{2}u(c_{1h}) + \frac{1}{2}u(c_{1l})$$
$$c_0 = 1 + qp$$
$$c_{1h} = (1 - d_h)(4 - p) + d_h(1.5)$$
$$c_{1l} = (1 - d_l)(2 - p) + d_l(1.5)$$
$$q = \frac{1}{2}(1 - d_h) + \frac{1}{2}(1 - d_l)$$

- Note the two new choice variables d_h and d_l that denote default in the high and low states, respectively

Two-Period Model: Incomplete Markets, Low Cost of Default

- To handle this problem with discrete choice, consider the cases separately.
- To shorten the process use the following insight: if you default in the H state you will definitely default in the L state because your income net of repayment in the H state is higher so incentive to default is lower
- The converse is not true: defaulting in the L state doesn't mean you default in the H state
- With this in mind, let's start with the guess that you default only if tomorrow's state is L

Two-Period Model: Incomplete Markets, Low Cost of Default

- Case 1: Set $d_l = 1, d_h = 0$. Then we know that

$$q = \frac{1}{2}, c_{1l} = 1.5$$

- Then we have given the optimality conditions

$$u'(c_0) = \lambda,$$
$$\frac{1}{2}u'(c_{1h}) = \mu_H$$
$$\frac{1}{2}\lambda = \mu_H$$

Two-Period Model: Incomplete Markets, Low Cost of Default

- These conditions imply $c_0 = c_{1h}$ which then yields

$$1 + \frac{1}{2}p = 4 - p \Rightarrow p = 2 \Rightarrow c_0 = c_{1h} = 2$$

- Then we need to make sure this is indeed an equilibrium
- An equilibrium is a fixed point: this means that the default decisions and prices we set at the beginning have to be consistent with the allocations obtained from those decisions
- Yes, this is the case: $d_l = 1$ is consistent with allocations as $2 - p = 0 < 1.5$; similarly $d_h = 0$ is consistent given that $c_{1h} = 4 - p = 2 > 1.5$

Two-Period Model: Incomplete Markets, Low Cost of Default

The equilibrium with incomplete markets and low default costs is given by

$$d_h = 0$$
$$d_l = 1$$
$$q = 0.5$$
$$p = 2$$
$$c_0 = 2$$
$$c_{1h} = 2$$
$$c_{1l} = 1.5$$

Two-Period Model: Incomplete Markets, High Cost of Default

- Suppose instead we have higher default costs so that borrower income becomes 0.5 in the event of default
- The borrower's problem becomes

$$\max U = u(c_0) + \frac{1}{2}u(c_{1h}) + \frac{1}{2}u(c_{1l})$$

$$c_0 = 1 + qp$$

$$c_{1h} = (1 - d_h)(4 - p) + d_h(0.5)$$

$$c_{1l} = (1 - d_l)(2 - p) + d_l(0.5)q = \frac{1}{2}(1 - d_h) + \frac{1}{2}(1 - d_l)$$

Two-Period Model: Incomplete Markets, High Cost of Default

- It is not hard to check that in this case the equilibrium is one where you never default

- In particular, check that the following is internally consistent (as $c_{1h} > 0.5, c_{1l} > 0.5$)

$$\begin{aligned} d_h &= 0 \\ d_l &= 0 \\ q &= 1 \\ p &= 1 \\ c_0 &= 2 \\ c_{1h} &= 3 \\ c_{1l} &= 1 \end{aligned}$$

- The borrower doesn't default because the cost of default is too high

Three-Period Model: Incomplete Markets with Default

- Three periods $t = 0, 1, 2$, and borrower's preferences are linear over consumption in each period
- Assume $\beta < 1$ so borrower prefers to frontload consumption
- Period 0 income is zero, periods 1 and 2 income can take on one of two values y_h or y_l according to

$$Pr(y_1 = y_H) = \alpha$$
$$Pr(y_2 = y_H | y_1 = y_H) = p_H$$
$$Pr(y_2 = y_H | y_1 = y_L) = p_L$$

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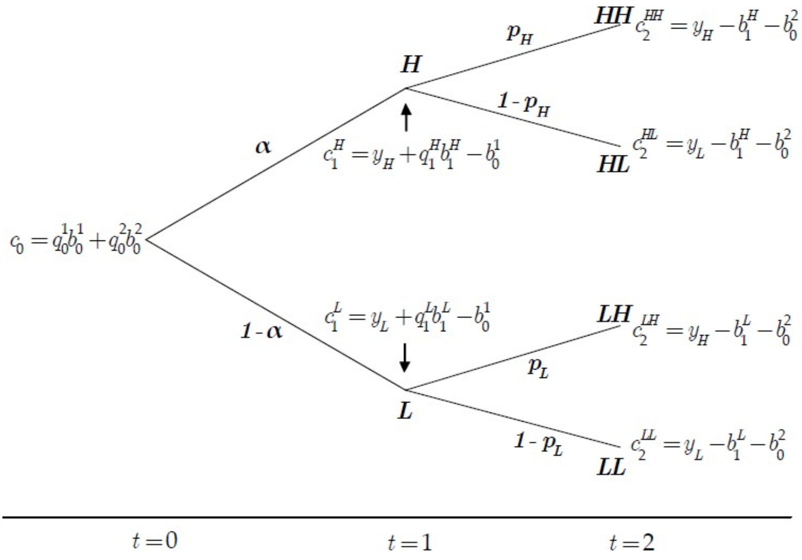
Three-Period Model: Incomplete Markets with Default



Three-Period Model: Incomplete Markets with Default

- The borrower's problem can be written as

$$\max U = E[c_0 + \beta c_1 + \beta^2 c_2]$$
$$c_0 = q_0^1(b_0^1, b_0^2)b_0^1 + q_0^2(b_0^1, b_0^2)b_0^2$$
$$c_1^i = y_i + q_1^i(b_1^i)b_1^i - b_0^1, \quad i = L, H$$
$$c_1^{ij} = y_j - (b_1^i + b_0^2), \quad j = L, H$$

- Notice the main difference between this setup and the one before: two types of bonds
- The short bonds b_0^1 and b_1^i and long bond b_0^2

Three-Period Model: Incomplete Markets with Default

- For a borrower that is sufficiently impatient to borrow enough that default happens but sufficiently patient to value the future resources he loses from default
- This means that he default after two L realizations of output, but will repay both types of debt otherwise
- This gives rise to the following bond prices

$$q_0^1 = 1$$
$$q_0^2 = \alpha + (1-\alpha)p_L$$
$$q_1^H = 1$$
$$q_1^L = p_L$$

Three-Period Model: Incomplete Markets with Default

- From frontloading, we know that consumption allocations are (note: corner, not interior, solution!)

$$
\begin{aligned}
c_0 &= y_L + p_L(y_H - y_L) + [\alpha + (1-\alpha)p_L]y_L \\
c_1^H &= (y_H - y_L)(1 - p_L), c_1^L = 0 \\
c_2^{HH} &= y_H - y_L, c_2^{HL} = 0 \\
c_2^{LH} &= 0, c_2^{LL} = y_{def} = 0
\end{aligned}
$$

Three-Period Model: Incomplete Markets with Default

- With only long-term debt, the allocations are

$$
\begin{aligned}
b_0^2 &= y_H \\
b_1^L &= 0, b_1^H = 0 \\
c_0 &= [\alpha p_H + (1-\alpha) p_L] y_H \\
c_1^H &= y_H, c_1^L = y_L \\
c_2^{HH} &= 0, c_2^{HL} = y_{def} = 0 \\
c_2^{LH} &= 0, c_2^{LL} = y_{def} = 0
\end{aligned}
$$

- In this case, you default in both HL and LL states and can only transfer resources from period 2 to period 0

Three-Period Model: Incomplete Markets with Default

- With only short-term debt, the allocations are

$$
\begin{aligned}
b_0^2 &= y_L + p_L y_H \\
b_1^L &= y_H, b_1^H = y_L \\
c_0 &= y_L + p_L y_H \\
c_1^H &= (1 - p_L) y_H, c_1^L = 0 \\
c_2^{HH} &= y_H - y_L, c_2^{HL} = 0 \\
c_2^{LH} &= 0, c_2^{LL} = y_{def} = 0
\end{aligned}
$$

- In this case, you default only in LL but can only transfer resources from period 2 to period 1, and from period 1 to 0

Arellano Ramanarayanan (2012)

- Short-term debt uses the threat of punishment (default loss) to enforce repayment and hence allows the borrower to transfer resources from period 1 to period 0; attempting to substitute LT debt for ST debt induces a fall in bond prices and undoes the motive to shift more resources to period 0

- Long-term debt provides a hedge against uncertainty in period 1 and hence allows the borrower to borrow more in the low state; to wit (given that $b_1^H = y_L - b_0^2$ and $b_1^L = y_H - b_0^2$:

$$c_1^H = y_H - b_0^1 + (y_L - b_0^1)$$
$$c_1^L = y_L - b_0^1 + p_L(y_H - b_0^1)$$

  Hence decreasing b_0^1 in favor of increasing b_0^2 reduces the PV of borrower's debt obligations in L but leaves it unchanged in H allowing him to essentially transfer resources from H to L (which ST can't do if he attempts to use it to substitute for LT debt)

A Brief Word on Multiplicity

- Consider a Laffer curve: to get revenues of 100, either sell 20 bonds at 5 dollars or 5 bonds at 20 dollars apiece
- A class of models consider the key variable to be fiscal or primary surplus (PS)
- There it is natural to expect multiplicity: if borrowers are concerned only with borrowing revenue, lenders' expectations become key to equilibirum selection

A Brief Word on Multiplicity

- In particular, if lenders expect borrowers to default, they charge a high price which then induces a default (self-fulfilling crises); if lenders are optimistic and charge a lower price instead, borrowers don't default in high output states of the world
- Uniqueness in the models we've looked at intuitively comes about because borrower maximizes consumption (not PS) and wouldn't choose the bad equilibrium as it means lower consumption tomorrow (for the same consumption today) due to output costs from default

Fundamental and Non-fundamental Risk

- Fundamental risk: risk of defaulting due to low output (fundamentals) and too much debt coming due, irrespective of lenders' expectations (output-driven default)
- Non-fundamental risk: risk due to investors expecting the government to default, raising the rate at which these bonds are sold and hence inducing default (expectations-driven default)
- See Corsetti and Dedola, Cole and Kehoe, Calvo

Corsetti and Dedola (2014)

- Show that multiplicity can be resolved with monetary backstop
- Key idea is that monetary backstop allows the central bank to issue nominal assets on which lenders expect no haircut or default (in contrast to government debt
- The paper characterizes the restrictions that limit the scope of this ability of the government to purchase debt (it has to be, in particular, a credible strategy that lenders will believe)

Bocola and Dovis (2015)

- Model with both long and short-term debt + rollover risk a la Cole and Kehoe
- Key point: use information embedded in comovement between interest rate spreads and debt duration to identify fundamental vs. non-fundamental risk
- Intuition: with fundamental risk, you want to lengthen debt duration to hedge against being unable to repay in state with low output tomorrow; with non-fundamental risk, you prefer short debt because it signals that you as a borrower intend to repay your debt soon ("creates commitment")
- Hence positive correlation between spreads and duration with fundamental risk, negative correlation between spreads and duration with non-fundamental risk

Conclusion

- In this lecture, we solved some simple finite-period models of sovereign default
- We find that both default and debt of different maturities allow borrowers to transfer resources both across states and over time
- This is the sense in which defaultable debt can be used to create claims that are state contingent (Perri 2008)
- Debt duration and multiple equilibria are also closely intertwined and active areas of frontier research