INTERMEDIATE MICROECONOMICS
ECON 3101, SECTION 002
MATH REVIEW

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1. Sets

Definition 1.1. A **set** is a collection of elements.

Examples

(a) $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ - the set of natural numbers between 1 and 11
(b) $\mathbb{R}$ - the set of real numbers
(c) $\mathbb{R}_+$ - the set of non-negative real numbers (NOTE: $0 \in \mathbb{R}_+$)
(d) $\mathbb{Z}$ - the set of integers
(e) $\mathbb{N}$ - the set of natural numbers (NOTE: $0 \in \mathbb{N}$)

Definition 1.2. A **Cartesian Product** of two sets A and B is a set of ordered pairs (a, b) such that $a \in A$ and $b \in B$. It is denoted as $A \times B$

Examples

(a) $\mathbb{R}^2 \equiv \mathbb{R} \times \mathbb{R}$ - Euclidean plane
(b) $\mathbb{R}^2_+$ - non-negative orthant of the Euclidean plane
(c) $\mathbb{R}^3$ - 3-dimensional Euclidean space (the upper bound for human imagination - usually)

I have borrowed these notes from Jacek Rothert with permission. They are based on the notes available on Summers Xia's website. However, there might be certain mistakes in these notes, for which I am solely responsible. I will be grateful for any corrections or suggestions.

Definition 1.3. A set A is **convex** if for any two elements a_1 and a_2 of the set and for any real number λ between 0 and 1, $\lambda a_1 + (1-\lambda)a_2$ is also an element of A.

Examples

(a) $\mathbb{R}$ - the set of real numbers is convex
(b) $[0,1]$ - the set of number between 0 and 1 is convex
(c) $[0,1) \cup (1,2]$ - the set of numbers between 0 and 2 excluding 1 is not convex. For example, take 0 and 2 as elements of the set and take $\lambda = 0.5$. Then $\lambda \cdot 0 + (1-\lambda) \cdot 2 = 0.5 \cdot 2 = 1$ which is not in the set.

2. Functions

Definition 2.1. A function f is a mapping from some set A (called the domain of the function) to some set B (called the range of the function) which satisfies a condition that to every element in A, f assigns a **unique** element from the set B. In short we write: $f : A \to B$

2.1. **Inverse functions.** We will encounter inverse function when we are talking about supply and demand. We can express price as a function of quantity and quantity as a function of price. Here is a formal definition.

Definition 2.2. Suppose that we are given a function $f : X \to Y$. We say that f has an inverse if there is a function g such that the domain of g is the range of f and such that

$$f(x) = y \text{ if and only if } g(y) = x$$

for every $x \in X$ and for every $y \in Y$.

Example

(a) Supply function specifies what the quantity supplied will be for a given price, e.g.

$$Q_s(p) = -100 + 5p$$

Inverse supply function tells us what price producer has to charge in order to supply a given quantity, e.g.

$$P_s(q) = 20 + \frac{1}{5}q$$

NOTE: you can easily verify that P_s is the inverse function of Q_s.

2.2. **Concave and convex functions.** This section will be a about real-valued functions.

Definition 2.3. Assume we have a function $f : A \to \mathbb{R}$, where A is a convex set. We say that the function f is **convex** if for all $a_1 \in A$, for all $a_2 \in A$ and for all $\lambda \in (0, 1)$ we have:

$$f(\lambda a_1 + (1 - \lambda)a_2) \leq \lambda f(a_1) + (1 - \lambda)f(a_2)$$

We say that the function f is **concave** if for all $a_1 \in A$, for all $a_2 \in A$ and for all $\lambda \in (0, 1)$ we have:

$$f(\lambda a_1 + (1 - \lambda)a_2) \geq \lambda f(a_1) + (1 - \lambda)f(a_2)$$

Examples

(a) Function $f(x) = x^2$ is convex. To see this take for example $x_1 = 0$, $x_2 = 2$ and $\lambda = \frac{1}{4}$. Then check that the inequality holds. (NOTE: it will really work for any x_1, x_2, λ)

(b) Function $f(x) = \sqrt{x}$ is concave. Try the same numbers as above. Try different numbers.

2.3. **Monotone functions.**

Definition 2.4. A real-valued function f of a real variable is **increasing** if $x_1 > x_2$ implies that $f(x_1) \geq f(x_2)$. It is **strictly increasing** if $x_1 > x_2$ implies that $f(x_1) > f(x_2)$.

Definition 2.5. A real-valued function f of a real variable is **decreasing** if $x_1 > x_2$ implies that $f(x_1) \leq f(x_2)$. It is **strictly decreasing** if $x_1 > x_2$ implies that $f(x_1) < f(x_2)$.

Definition 2.6. A real-valued function f of a real variable is **monotone** if it is either increasing or decreasing.

Remark 2.7. A constant function is both increasing and decreasing (but not strictly).

3. Calculus

This section will summarize the information you would need to know about derivatives of real-valued functions of one variable (i.e. $f : \mathbb{R} \to \mathbb{R}$).

Definition 3.1. Derivative of a real-valued function of one variable at a point x^* is defined as:

$$f'(x^*) \equiv \lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h}$$

Definition 3.2. A real-valued function is differentiable at x^* if the above limit exists.

Definition 3.3. A real-valued function is differentiable if it is differentiable at every point in its domain.

Remark 3.4. Derivative is a number.

Remark 3.5. The following notations are equivalent:

- $f'(x^*)$
- $\frac{d}{dx} f(x^*)$
- $\frac{d}{dx} f(x)|_{x=x^*}$

Intuitively, the derivative measures how a very very very small rise in the argument of the function will increase the value of the function.

Example
Suppose we have a cost function of a firm given by a formula:

$$c(q) = 5 + q^2.$$

The intuition behind such a cost function is that there is a fixed cost of 5 and a variable cost $vc(q) = q^2$. The derivative of the cost function with respect to q is:

$$c'(q) = 2q.$$

$c'(q)$ is the marginal cost (MC). In ECON 1101 we were told that MC measures how the total cost changes if the output increases by 1 unit. Here we don't have a unit increase in q but an infinitely small increase. It may seem less intuitive economically but turns out to be much more tractable and easier to work with.

Geometric interpretation

Derivative at a point equals the slope of the graph of the function at that point.

Derivative as a function can be understood as follows. We say that a function f' is a derivative of a function f if for every x in the domain of f' the following is true:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

3.1. **Important rules for differentiation.** Here are some important rules for differentiation (in all examples k is constant different from 0).

(a) Derivative of a constant

$$\frac{dk}{dx} = 0$$

(b) Multiplication by a constant

$$\frac{df(k \cdot x)}{dx} = k \cdot \frac{df(x)}{dx}$$

(c) Derivative of a sum / difference

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$$

(d) Chain rule

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

(e) Product rule

$$\frac{d}{dx} [f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$$

(f) Quotient rule

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

(g) Power function

$$\frac{d}{dx} x^n = nx^{n-1}$$

(h) Exponential function

$$\frac{d}{dx} e^x = e^x$$

(i) Logarithmic function (with base e)

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

3.2. **Derivatives and monotonicity.** Knowing the sign of the derivative of the function we can draw some conclusions about the behavior of the function. The following theorem sheds some light on monotonicity.

Theorem 3.6. *Suppose that we have a real-valued function f defined on the set of real numbers and suppose that f is differentiable. Suppose that $f' > 0$. Then the function is strictly increasing. Similarly if $f' < 0$ then the function is strictly decreasing.*

3.3. **Second derivative.**

Definition 3.7. Second derivative of a function f at x^* is defined as:

$$f''(x^*) \equiv \lim_{h \to 0} \frac{f'(x^* + h) - f'(x^*)}{h}$$

NOTE: second derivative is simply a derivative of the derivative.

Examples

(a) $f(x) = x^2$, then $f'(x) = 2x$ and $f''(x) = 2$
(b) $f(x) = e^x$, then $f'(x) = e^x$ and $f''(x) = e^x$
(c) $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$

3.3.1. *Second derivative and concave/convex functions.* Second derivative can be used to determine whether a real-valued function of one variable is concave or convex.

Theorem 3.8. *Suppose that f is a real-valued function of one variable and suppose that f'' exists. Then f is convex if and only if $f'' \geq 0$.*

Theorem 3.9. *Suppose that f is a real-valued function of one variable and suppose that f'' exists. Then f is concave if and only if $f'' \leq 0$.*

4. Functions of more than one variable

In economics we usually encounter functions of more than one variable. Consumer's happiness depends on more than just one factor. It may depend on how many candy bars he can eat per day, how much time he spends with friends and how long he has to study. This would make consumer's happiness a function of three variables. Some more examples follow.

Examples

(a) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ and for all $(x_1, x_2) \in \mathbb{R}^2$ $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$. Function f measures the distance of the point on the plane from the origin.
(b) Suppose that $f : [0, 24] \times \mathbb{R}_+ \to \mathbb{R}_+$ and for all $h \in [0, 24]$ and for all $w \in \mathbb{R}_+$ $f(h, w) = h \cdot w$. We can interpret h to be the number of hours a person works in a day and w to be the hourly wage. Then the function f calculates daily earnings of that person.

4.1. **Multivariate Calculus.** This section will cover basics of the multivariate calculus.

Definition 4.1. Let $y = f(x_1, x_2, ..., x_i, ..., x_n)$. A **partial derivative** of f with respect to x_i at $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_i, ..., \hat{x}_n)$ [1] is:

$$\left.\frac{\partial f(x)}{\partial x_i}\right|_{\mathbf{x}=\hat{\mathbf{x}}} \equiv \lim_{h\to 0} \frac{f(\hat{x}_1, \hat{x}_2, ..., \hat{x}_i + h, ..., \hat{x}_n) - f(\hat{x}_1, \hat{x}_2, ..., \hat{x}_i, ..., \hat{x}_n)}{h}$$

NOTE: equivalent notations are $\frac{\partial y}{\partial x_i}$ or $f_i(\mathbf{x})$.

The intuition is that we keep the values of all other variables fixed (as if they were parameters) and calculate a regular variable of a real-valued function of one (in this case x_i) variable.

Definition 4.2. Let $y = f(\mathbf{x}) = f(x_1, x_2, ..., x_i, ..., x_n)$. A **total differential** of f is defined as:

$$df(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_1}dx_1 + \frac{\partial f(\mathbf{x})}{\partial x_2}dx_2 + ... + \frac{\partial f(\mathbf{x})}{\partial x_n}dx_n$$

The above definition says that a total change in the value of f can be decomposed into changes resulting from changes of $x_1, x_2, ..., x_n$. The symbol dx_i represents a change in the value of x_i and is assumed to be very very small.

Examples

(a) Suppose that we have a an earning function of an individual given by: $E(h, w) = h \cdot w$. It's a function of two variables: hours worked (h) and hourly wage (w). We know that:

$$\frac{\partial E}{\partial h} = w \text{ and } \frac{\partial E}{\partial w} = h$$

Hence the total differential will be:

$$dE = \frac{\partial E}{\partial h}dh + \frac{\partial E}{\partial w}dw = w \cdot dh + h \cdot dw$$

[1] The boldface letters denote vectors.

(b) This example will use the chain rule. Suppose that firm's profit is a function of output and advertising. Let q denote output and a denote advertising (it can be for example the amount of \$ spent on it). The profit function is: $\pi(q, a) = 10q - q^2 + 2qa + 10a - a^2$. We also know that firm's output is a function of world price p, i.e. $q(p) = 50 + 2p$. Initially $q = 5$, $a = 5$. Calculate the change in profit if the world price falls by 0.1 and firm decides to increase the money spent on advertising by 0.1.

ANSWER: we compute the total differential of the profit function. It is given by:

$$d\pi = \frac{\partial \pi}{\partial q} dq + \frac{\partial \pi}{\partial a} da = \frac{\partial \pi}{\partial q} \frac{\partial q}{\partial p} dp + \frac{\partial \pi}{\partial a} da =$$

$$= (10 - 2q + 2a) 2dp + (2q + 10 - 2a) da$$

Since $dp = -0.1$ and $da = 0.1$ we get:

$$d\pi = (10 - 2 \cdot 5 + 2 \cdot 5) \cdot 2 \cdot (-0.1) + (2 \cdot 5 + 10 - 2 \cdot 5) \cdot 0.1 = -20 \cdot 0.1 + 10 \cdot 0.1 = -1$$

5. Optimization methods

In economics we are interested in problems like (i) consumer maximizing utility, (ii) firm maximizing profit or (iii) minimizing cost. Finding solutions to maximization and minimization problems becomes much easier once we can apply calculus techniques.

5.1. Unconstrained optimization with one variable.

Definition 5.1. Function f has a **local maximum** at x^* if there exists some open interval I that contains x^* and $f(x^*) \geq f(x)$ for all $x \in I$.

Definition 5.2. Function f has a **global maximum** at x^* if $f(x^*) \geq f(x)$ for all x in the domain of f.

Definition 5.3. If a function f has a local (global) maximum at x^* then x^* is called a local (global) maximizer of f. We write it as $x^* = \text{argmax } f(x)$.

Next theorem says that if a differentiable function has a local maximum/minimum at a point, then the derivative at this point has so be zero.

Theorem 5.4. ***First Order Necessary Condition for maximum/minimum***
Let f be a real-valued function of one variable. Suppose that f has a local maximum/minimum at some point x^* and suppose that $f'(x^*)$ exists. Then $f'(x^*) = 0$.

It is important to realize what the above theorem **doesn't** say. It does **not** say that if $f'(x^*) = 0$ then f has a maximum/minimum at the point. In particular, we are not able to distinguish between maximum and minimum. It can also happen that neither one occurs.

The next two theorems help us distinguish between maximum and minimum.

Theorem 5.5. ***Second Order Necessary Condition for maximum***
Let f be a real-valued function of one variable. Suppose that f has a local maximum at some point x^* and suppose that $f''(x^*)$ exists. Then $f''(x^*) \leq 0$.

Theorem 5.6. ***Second Order Necessary Condition for minimum***
Let f be a real-valued function of one variable. Suppose that f has a local minimum at some point x^* and suppose that $f''(x^*)$ exists. Then $f''(x^*) \geq 0$.

It is still possible however that we have neither a maximum nor minimum. For example, consider a function $f(x) = x^3$. We know that $f'(0) = 0$ and $f''(0) = 0$, so we have that both $f''(0) \geq 0$ and $f''(0) \leq 0$. We can see that in this case neither maximum nor minimum exists.
Fortunately we have sufficient conditions for maximum and for minimum.

Theorem 5.7. ***Sufficient Condition for maximum***
Let f be a real-valued function of one variable. Suppose that $f''(x^*)$ exists and suppose that $f'(x^*) = 0$ and $f''(x^*) < 0$. Then f has a local maximum at x^*.

Theorem 5.8. ***Sufficient Condition for minimum***
Let f be a real-valued function of one variable. Suppose that $f''(x^*)$ exists and suppose that $f'(x^*) = 0$ and $f''(x^*) > 0$. Then f has a local minimum at x^*.

Examples

(a) Consider a function $f(x) = 16x - 2x^2$. We can verify that the function has a local maximum at a point $x^* = 4$. We have that $f'(4) = 16 - 4x\big|_{x=4} = 16 - 4 \cdot 4 = 0$. FONC is satisfied. We can calculate $f''(4) = -4 < 0$, hence the sufficient condition is satisfied. Function f has a local maximum at 4.

(b) This example will highlight the difference between a local and a global maximum/minimum. Suppose that we have a function $f : \mathbb{R}_+ \to \mathbb{R}$, i.e. f is a real-valued function defined on the set of non-negative real numbers. Assume $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 10$. We want to find local and global maximuma/minima. FONC for maximum/minimum is: $f'(x) = x^2 - 3x + 2 = 0$, which can be solved for x to give $x = 1$ or $x = 2$. Now we notice that $f''(x) = 2x - 3$, hence $f''(1) = -1 < 0$ and $f''(2) = 1 > 0$, so there is a local maximum at $x = 1$ and a local minimum at $x = 2$. We can evaluate the function at $x = 1$ and at $x = 2$: $f(1) = 10\frac{5}{6}$ and $f(2) = 10\frac{4}{6}$. However, $f(0) = 10 < 10\frac{4}{6} = f(2)$, so global minimum is not at $x = 2$ (it's actually at x=0). We also have that $\lim_{x\to\infty} f(x) = \infty$, so the function doesn't have global maximum.

5.2. **Unconstrained optimization with two variables.** Now assume that we have a real-valued function $y = f(x_1, x_2)$, where $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$, so $f : \mathbb{R}^2 \to \mathbb{R}$.

Theorem 5.9. ***First Order Necessary Condition for maximum/minimum***
Suppose that f has a local maximum/minimum at some point $x^* = (x_1^*, x_2^*)$. Suppose that $\frac{\partial f(x^*)}{\partial x_1}$ and $\frac{\partial f(x^*)}{\partial x_2}$ exist. Then $\frac{\partial f(x^*)}{\partial x_1} = \frac{\partial f(x^*)}{\partial x_2} = 0$.

Theorem 5.10. ***Sufficient Conditions for maximum***
Suppose that f is twice differentiable at . Suppose that $\frac{\partial f(x^*)}{\partial x_1}$ and $\frac{\partial f(x^*)}{\partial x_2}$ exist. Then $\frac{\partial f(x^*)}{\partial x_1} = \frac{\partial f(x^*)}{\partial x_2} = 0$.

5.3. **Constrained optimization.** This section is very important for our course. We will be mostly dealing with constraint maximization/minimization problems. Consider the following example.

Example
Suppose you have 20 feet of rope and you need to construct a rectangular frame with maximal area. This is a problem of constrained maximization.
Solution: First, we write this problem in a precise mathematical language:

$$\max_{x,y} f(x,y) = x \cdot y$$

s.t.

$$2x + 2y = 20$$

To solve this problem means to find (x^*, y^*) that maximize the value of f under the constraint that $2x^* + 2y^* = 20$. f is called the **objective** function. In our case f represents the area of the rectangular, with the sides x and y. Under the max we write the **choice** variables. In our case these are the lengths of the sides. s.t. means such that or subject to. After s.t. follows the constraint, which in our case means that the sum of lengths of all sides is equal to 20 feet. It is very important in this course to be able to write the problems in a clear way, such that a mathematician, who doesnt know economics, will understand what is written.

There are two ways to solve this problem.

(a) Substitution. We can rewrite the constraint as $y = 10 - x$. Then $f(x, y) = x(10 - x) = 10x - x^2$. This becomes an unconstrained maximization problem for a function of one variable. Using previously learned techniques we obtain the solution: $x^* = 5$ (check that both FONC and sufficient condition are satisfied) and since $2x^* + 2y^* = 20$ we get $y^* = 5$.

(b) Lagrange multipliers. This method deserves a separate subsection.

5.3.1. *Lagrange multipliers.* I will start this section with a simplified, two-variable version of the theorem which we will use to solve constrained optimization problems.

Theorem 5.11. *Let f and g be two real-valued continuously differentiable functions of two variables. Suppose that (x_1^*, x_2^*) is a solution to the following maximization problem:*

$$\max_{x_1, x_2} f(x_1, x_2)$$

$$s.t.$$

$$g(x_1, x_2) \quad = \quad c$$

*where c is a real number (constant). Suppose also that (x_1^*, x_2^*) is not a critical point [2] of g. Then there exists a real number λ^* (called the **Lagrange Multiplier**) such that $(x_1^*, x_2^*, \lambda^*)$ is a critical point of the following function (usually called a **Lagrangian**):*

$$L(x_1, x_2, \lambda) \equiv f(x_1, x_2) + \lambda \cdot [c - g(x_1, x_2)],$$

i.e.

$$\frac{\partial L(x_1^*, x_2^*, \lambda^*)}{\partial x_1} = 0,$$

[2] Critical point of a function is the value of the argument where all partial derivatives are zero.

$$\frac{\partial L(x_1^*, x_2^*, \lambda^*)}{\partial x_2} = 0,$$

and

$$\frac{\partial L(x_1^*, x_2^*, \lambda^*)}{\partial \lambda} = 0.$$

The above version is probably the simples possible. There are generalizations of the theorem to arbitrary number of variables, as well as equality and/or inequality constraints [3].

Let's apply the Lagrange Theorem to our example with rope and rectangular box. The objective function is $f(x_1, x_2) = x_1 \cdot x_2$, the constraint function is $g(x_1, x_2) = 2x_1 + 2x_2$ and constant $c = 20$. The Lagrangian function is:

$$L(x_1, x_2, \lambda) = x_1 \cdot x_2 + \lambda[20 - 2x_1 - 2x_2]$$

First Order Necessary Condition is (from the Lagrange Theorem) that all partial derivatives of the Lagrangian are zero, i.e.:

$$\frac{\partial L(x_1^*, x_2^*, \lambda^*)}{\partial x_1} = x_2^* - 2\lambda^* = 0 \tag{5.1}$$

$$\frac{\partial L(x_1^*, x_2^*, \lambda^*)}{\partial x_2} = x_1^* - 2\lambda^* = 0 \tag{5.2}$$

$$\frac{\partial L(x_1^*, x_2^*, \lambda^*)}{\partial \lambda} = 20 - 2x_1^* - 2x_2^* = 0 \tag{5.3}$$

Note that equation (5.3) simply says that the constraint in the maximization problem has to hold. The above system of 3 equations and 3 unknowns ($x_1^*, x_2^*,$ and λ^*) is quite easy to solve (do it) and we get:

$$x_1^* = 5$$
$$x_2^* = 5$$
$$\lambda^* = 2.5$$

The Lagrange Multiplier λ^* has a nice economic interpretation. It tells us how much will the value of our maximized objective function increase if we relax our constraint by 1. Suppose we will increase the length of

[3] Version with inequality constraints is the Kuhn-Tucker theorem

our rope by 1 foot, so that it's now 21 feet. The value of $\lambda^* = 2.5$ says that the maximum possible area of the rectangular constructed from this rope will increase by 2.5 square feet. Since for the 20 foot rope the maximal area was 25, it should now be approximately 27.5. Please check it yourself (you will find that it will not be exactly 27.5. Why?).